Insight from Geometry and Physics into the Construction of Egyptian Old Kingdom Pyramids

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When confronted with a new problem in physics or engineering, the physicist or engineer often makes 'back-of-the-envelope' calculations. These calculations explore the problem using mathematical and physical principles, known facts, and estimates of unknown values where necessary. The intent is to determine constraints for or estimates of the expected results of more detailed investigations. Often the calculations reveal preliminary results, and identify areas which require additional information. Sometimes fortune smiles and unexpectedly detailed and useful insight is gained with little effort.

This report applies such an approach to construction of Egyptian Old Kingdom pyramids. The aim is to determine what can be learned about pyramid construction, without any reference to particular construction practices. We shall proceed as far as possible strictly from principles of physics and geometry and from a few simple assumptions. Simple considerations of size, mass, and shape, and of time available for construction yield surprisingly detailed estimates for rates of construction. Less well-determined but useful estimates of manpower also emerge.

Pyramids offer an unusual situation in architectural investigations. Their extremely simple shape, and the simplicity of the construction, mean that arguments from geometry and physics can be properly and easily applied. Unfortunately this approach is not helpful for more common and more complex architectural remains.

Important details about the construction of pyramids built during the Old Kingdom of Egypt are unknown to us. For example, no reliable accounts survive of methods of construction, or of the number of men engaged in building the pyramids. Two thousand years after the pyramids were built, Herodotus was told that some 100,000 men were employed in the construction of the Great Pyramid. Unfortunately much that Herodotus reported about ancient Egypt is now known to be unreliable.

For purposes of illustration, this report uses the pyramid of Khufu at Giza, for the reason that more detailed information is available for that pyramid than for any other. The same principles apply to all the solid masonry pyramids of the Old Kingdom, and some basic findings are presented for all the surviving large pyramids. For mathematical consistency results are presented to three significant figures, though the available information does not support anywhere near the implied precision. Khufu's pyramid at Giza was the fifth large masonry pyramid completed in the Old Kingdom of Egypt. It was constructed apparently during the twenty-sixth century BC. Earlier large pyramids had been under construction for some 80 years before work began on this pyramid (Edwards 1988). Khufu's pyramid benefited from this period of increasing sophistication in pyramid construction (Fakhry 1961; Edwards 1988; Arnold 1991). Two more large pyramids were built at Giza after Khufu's pyramid (Fig. 1).

Though somewhat damaged, Khufu's pyramid is still largely intact, and its dimensions are well established. The base, which was nearly a perfect square, has sides which originally averaged 230.4 m in length. The pyramid was 146.7 m high (Edwards 1988). From these values one computes an original volume of 2.59×10^6 m^3 (one third of base area times height). The mass of the finished edifice was 7.01×109 kg (volume times density), taking the average density of the pyramid as 2.70×10^3 kg m^{-3}.

Khufu's reign is generally accepted to have lasted 23 years. We do not know for sure, however, if that figure is correct, nor do we know how much of his reign was occupied in construction of his pyramid. One may reasonably suppose the ancient builders had no idea how long the reign would last, and it is unlikely that pyramid construction coincided exactly with the duration of the reign. Lacking better guidance, and to make the simplest possible assumption, we assume construction of this pyramid required 23 years. If construction continued year-round, some 8400 days were available to assemble the pyramid, and this figure is used in most of the calculations presented here. Lacking knowledge of the actual length of time spent in constructing each pyramid means it is pointless, for the purpose of these calculations, to seek highly accurate values for other parameters in calculations where the construction time is a factor.

The only assumptions about pyramid construction techniques used in this report are that pyramids were made of stone blocks quarried, shaped, moved, and installed by human power alone, that the pyramids were largely built in layers from bottom to top, and that horizontal transport (but not necessarily vertical lifting) was effected by dragging the blocks on wood sleds, for which there is considerable evidence in other building contexts (Clarke & Engelbach 1930; Arnold 1991). We treat a pyramid as if it were a

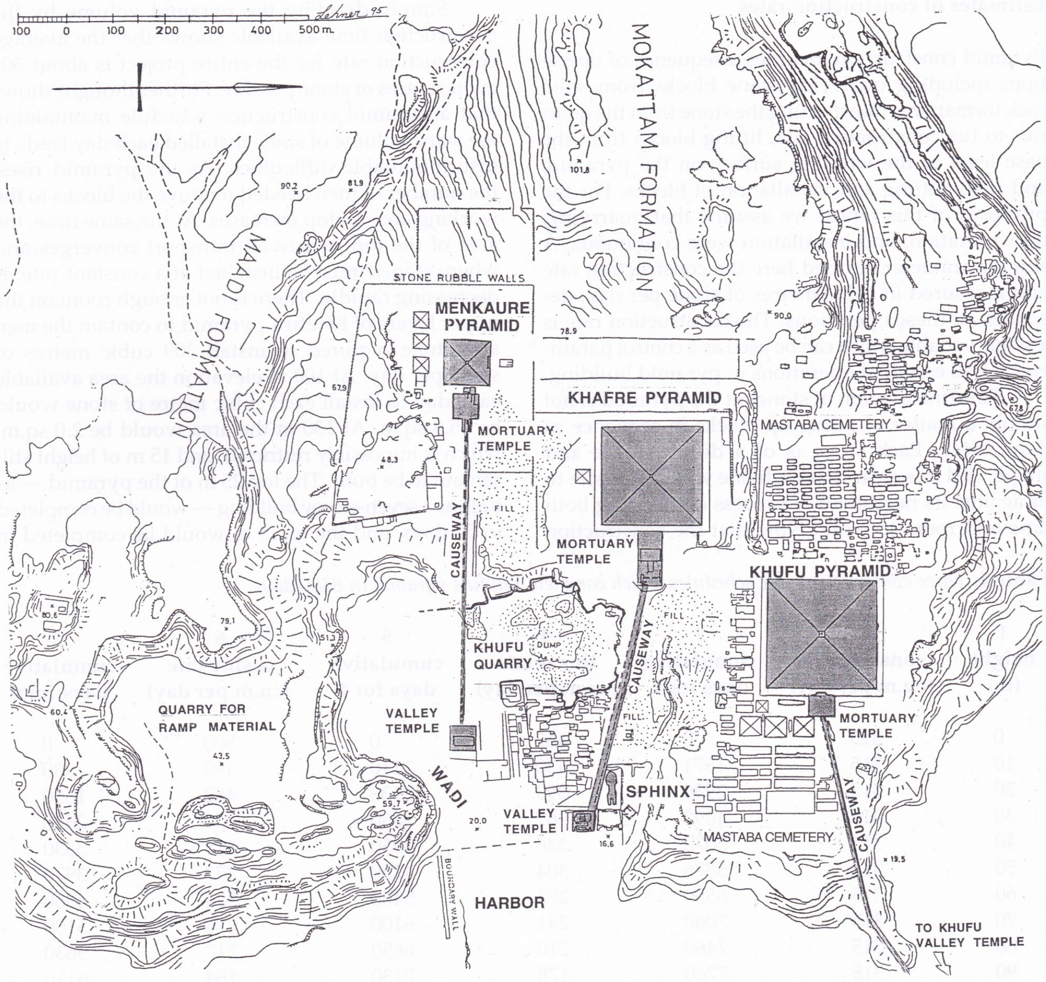

Figure 1. *Map of the Giza area, showing the location and size of the pyramids and related structures, and the location of the quarry for Khufu's pyramid. Khufu's pyramid was the first constructed in this location.*

homogeneous structure of locally-quarried limestone. The details of construction, such as internal passages and chambers lined with granite imported from Aswan, and the external casing of Tura limestone, certainly added important demands to pyramid construction, but are outside the scope of this kind of inquiry, and will not modify the basic conclusions found here.

Estimates of construction rates

Pyramid construction includes a sequence of operations including quarrying stone blocks from solid rock formations, transporting the stone from the quarries to the construction site, lifting blocks from the base level to the working surface on the pyramid, and final cutting and installation of blocks. For the purposes of this report we assume that quarrying, transportation, and installation were continuous. A single parameter, termed here the construction rate and measured in cubic metres of stone per day, describes all these operations. The construction rate is a single variable which can be used as a control parameter for all essential operations in pyramid building.

Final installation of stones at the pyramid is not simply a matter of piling up blocks in sequence as they arrive. Each stone is of a different size and shape, and it is clear that each one was cut on site to fit against its neighbour, a process demanding both time and room to work. Note that the construction site is on top of the pyramid, a region which rapidly shrinks in area as the pyramid rises.

The shape and size of the pyramid, combined with the estimate of construction time, and without any other information, reveal surprisingly detailed information about feasible schedules of construction rate as the pyramid was erected. These conclusions are the most accurate and the best supported of this report.

Simply dividing the pyramid volume by the construction time available shows that the average construction rate for the entire project is about 309 cubic metres of stone per day. Further thought shows that a pyramid construction schedule maintaining the same volume of stone installed each day leads to insurmountable difficulties. As the pyramid rises, the number of men needed to move the blocks to the working area on top increases. At the same time, the area of the top, where all transport converges and where stones must be installed at a constant rate, is decreasing rapidly. There is not enough room on the upper layers of Khufu's pyramid to contain the men and stone required to install 309 cubic metres of stone per day. At 100 m elevation the area available each day to install each cubic metre of stone would be 15.8 sq.m. At 130 m the area would be 2.0 sq.m, which is impossibly restricted, and 15 m of height still remain to be built. The top 25 m of the pyramid — as high as a seven-storey building — would be completed in 52 days, and the top 5 m would be completed in

Table 1. *Three construction rate schedules which complete Khufu's pyramid in 8400 days.*

1 height (m)	2 constant rate (cu.m per day)	3 cumulative days for 2	4 linear rate (cu.m per day)	5 cumulative days for 4	6 cosine rate (cu.m per day)	7 cumulative days for 6
0	315	0	462	0	500	0
10	315	1571	430	1110	499	980
20	315	2930	399	2130	482	1850
30	315	4090	367	3080	454	2630
40	315	5070	336	3950	418	3330
50	315	5876	304	4750	373	3970
60	315	6530	273	5460	324	4560
70	315	7060	241	6100	270	5100
80	315	7460	210	6650	217	5630
90	315	7760	178	7130	164	6110
100	315	7970	147	7540	116	6580
110	315	8110	115	7860	74	7020
120	195	8200	84.1	8100	40	7450
130	76.3	8280	52.6	8270	16	7870
140	12.2	8360	12.2	8360	2.6	8260
145	0.78	8396	0.78	8396	0.78	8396

one day, if the constant rate could be maintained (divide the volume of these segments by the average construction rate). There simply would not be enough room for the men required to deliver and install the volume of material required each day, at the upper levels of the pyramid, if a constant rate were attempted. This conclusion applies to all Old Kingdom pyramids. The obvious solution is that increased rates of building were used at the lower levels, where plenty of space is available and where access is easy, permitting lower rates at higher levels where space is limited and where lifting stones is more demanding. Reduced construction rates near the top have little effect on overall pyramid construction, but they must be included in any realistic construction model.

To determine reasonable construction rate schedules, a simple computer program was written to add up the volume accumulated and the time spent to reach each level on the pyramid, using any given schedule of construction rate, under the constraint that the total time spent is 8400 days. The program also ensures that the area for installation never falls below 10 square metres per cubic metre of stone per day. The simplest construction rate schedule is a constant rate. Using that approach the construction rate schedule shown in Table 1 (column two) is determined. During most of the pyramid construction 315 cubic metres of stone is quarried and installed every day (Fig. 2, curve A). The adjacent column of Table 1 also shows the number of days required to reach any level on the pyramid. Owing to the restrictions in the upper levels, the construction rate drops rapidly above 110 metres pyramid height, after 8110 days (22 years). The schedule shown in Table 1 (columns two and three) is one possible way to build the pyramid, but it requires more men, compared to using a higher initial construction rate.

Another schedule that also completes Khufu's pyramid in 8400 days, is shown in column four of Table 1, and in Figure 2 (curve B). In this case, the construction rate declines directly in proportion to the increasing height of the pyramid. Several other variations of construction rate with height were tested. For example, the cosine function was used to make a smooth variation of construction rate with height which begins at a constant rate and then gradually declines (Table 1, column six & Fig. 2, curve C). These are mathematical investigations into what constitutes a feasible variation of construction rate with height. We do not propose that the Egyptians based scheduling on any of these particular functions, but simple considerations of geometry and practicality strongly indicate that any realistic schedule for construction of the pyramid must have fallen near the curves shown in Figure 2.

Estimates of manpower required for transportation of stone blocks

Any feasible construction rate schedule can be used to examine manpower required during construction. For example, if we knew the actual techniques used to move stone from the quarry to the pyramid, we would be in a position to estimate the workforce employed in transportation. Since the ancient techniques are not well known, other estimates of manpower, of somewhat less reliability, can be made by using simple principles of physics.

Manpower required for raising stone is estimated by calculating the potential energy. Potential energy is the minimum energy required to lift material above an arbitrary reference level. The potential energy of a simple pyramid, relative to its base, is $(g\rho W^2H^2)/12$, where g is the acceleration of gravity ($9.81\ m/s^2$), ρ is the average density of the pyramid ($2.70 \times 10^3\ kg/m^3$), W is the width of the base, and H is the height. The density chosen is typical of limestone. The potential energy of Khufu's pyramid, when new, was close to 2.52×10^{12} Joules, using the dimensions cited above. Dividing the potential energy of the pyramid by the average energy one workman can provide in one day gives an absolute minimum value of the number of man-days required to lift the material of the pyramid to the levels where it is found. Dividing this by the number of days available gives an absolute minimum estimate of the number of men required simply to lift the stones. This does not include inefficiencies in lifting.

Using the potential energy, and an estimate of 2.40×10^5 Joules (Hudson 1917) for the average amount of useful work provided by a man in a day, gives a minimum number of 1250 men, employed throughout the entire construction period (8400 days), just to raise the blocks to a level sufficient to accumulate the potential energy of the completed pyramid. After including a factor for inefficiency, one could arrive at a realistic estimate of the number of men required to provide the energy to raise the stones onto the pyramid. This computation is merely to demonstrate use of the potential energy to compute manpower required for lifting; it does not reveal much about pyramid construction other than the fact that the manpower required purely for lifting is quite feasible.

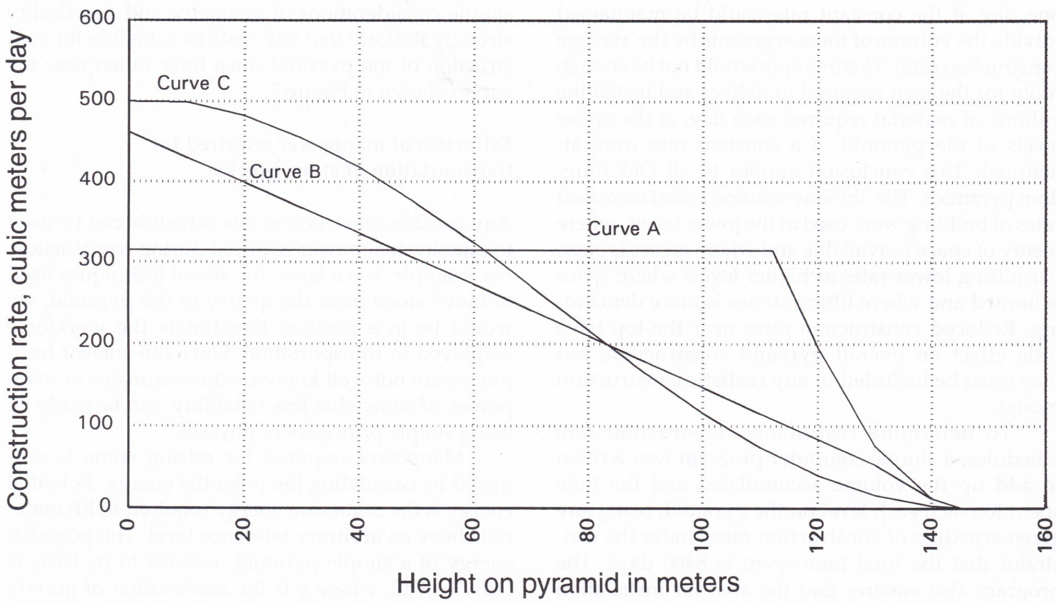

Figure 2a. *Three schedules of construction rates (cubic metres per day) with height on the pyramid which will complete Khufu's pyramid in 8400 working days. (Curve A, Table 1 column two; Curve B, Table 1 column four; Curve C, Table 1 column six.)*

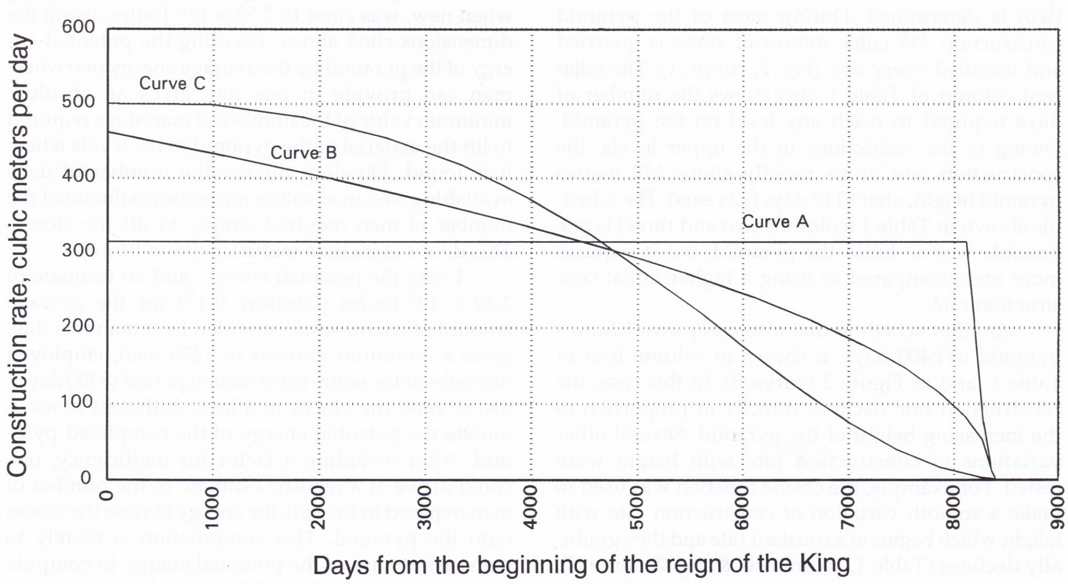

Figure 2b. *Three schedules of construction rates (cubic metres per day) with days in the King's reign which will complete Khufu's pyramid in 8400 working days. (Curve A, Table 1 columns two and three; Curve B, Table 1 columns four and five; Curve C, Table 1 columns six and seven.)*

The value quoted above for the amount of useful work or energy a man can provide in one day is not high. Indeed, I have seen young adolescents achieve this figure. Given the lack of accuracy in other terms used here, especially the construction time, there is little merit in debating the exact value of the average amount of work provided per man per day, for the purposes of the argument presented here.

Given a reasonable variation of construction rate with height of the pyramid, the manpower required for raising blocks to each level can be computed using the same principle. For an improved estimate we use as the reference level the floor of the adjacent large Central Field quarry (Lehner 1984, 150), 19.0 m below the base of the pyramid. The potential energy of all stone added each day at height h on the pyramid is $PE = g \times m(h + 19.0)$, where m is the mass of stone delivered to that level each day in kg, g is the acceleration of gravity (9.81 m/s^2), h is the height of the working level above the pyramid base in metres, and 19.0 m is the elevation of the pyramid base above the quarry floor. To compute the number of workmen required for lifting at any day or level on the pyramid, we compute m from the construction rate per day (cubic metres times the density), and divide the resulting PE by the average energy a man provides in one day.

The resulting manpower required for the lifting or vertical component of transportation is shown in Table 2 (column three). This was computed using the linear decrease of construction rate with height from Table 2 (column two copied from Table 1 column four). The number of men required for lifting reaches a maximum (2380) when the pyramid is 60 m high, and drops to small numbers near the top. Note that this manpower estimate for lifting is a minimum; it is based on the potential energy of the stones, but includes no extra factor for inefficiencies in lifting technique. As a check on calculations the cumulative total work accomplished by lifting was verified to be equal to the total potential energy of the completed pyramid.

At the same time, the manpower for moving stone from the quarry to the pyramid can be estimated using another simple physical principle. By separating the manpower demands for stone transportation from the quarry to the site of block installation into two components (horizontal transport and vertical lifting), questions of construction technique are circumvented. Separation of transportation into the two unrelated horizontal and vertical components is a mathematical technique to permit estimates of the work required based on fundamental physical principles. Actual division of transport into purely horizontal and vertical motions are not

Table 2. *Construction rate schedule and related manpower estimates to complete Khufu's pyramid in 8400 days.*

1 height on pyramid	2 construction rate	3 men for lifting	4 men for hauling	5 men using ramp	6 column 5 times 1.5	7 low manpower estimate	8 high manpower estimate
0	462	970	3230	4200	6310	9540	12,800
10	430	1380	3010	4390	6590	9600	12,600
20	399	1716	2794	4510	6770	9560	12,400
30	367	1990	2570	4560	6840	9410	12,000
40	336	2190	2350	4540	6810	9160	11,500
50	304	2320	2130	4450	6680	8810	10,900
60	273	2380	1910	4290	6440	8350	10,300
70	241	2370	1690	4060	6090	7780	9460
80	210	2290	1470	3760	5640	7110	8590
90	178	2150	1250	3400	5100	6350	7590
100	147	1930	1030	2960	4440	5470	6500
110	115	1640	810	2450	3680	4490	5300
120	84.1	1290	590	1880	2820	3410	4000
130	52.6	860	370	1230	1850	2220	2590
140	12.2	215	85	300	450	535	621
145	0.78	15	5	20	30	35	41

required for this mathematical technique to give good estimates. The total manpower, the sum of manpower determined from the horizontal and vertical components, is the meaningful result.

It seems certain that stones were moved from the quarry to the pyramid by dragging on wooden sleds on prepared paths (Clarke & Engelbach 1930; Arnold 1991). Work is required to overcome friction; its value is $g \times m \times C_f \times D$, where again g is 9.81 m/s^2, m is the mass of the stone (kg), C_f is the coefficient of friction (unitless), and D is the distance moved (m).

Two sets of evidence suggest the amount of friction the early workers encountered. Some surviving paths used in pyramid construction include wooden baulks sunk traversely into the path (Dunham 1956; Arnold 1991). The coefficient of sliding friction of dry wood on dry wood is about 0.2 (Hudson 1917). Just for example, if we assume a worker can exert a force of 112 Newtons (25 pounds), then he can pull a mass of 57 kg (125 pounds) on flat ground using wooden sleds on dry wooden paths. Since it is believed that the early Egyptians lubricated the surface with water or water mixed with clay, the friction would be reduced, perhaps by a factor of two or three, giving a coefficient of friction of 0.10 to 0.07. In the example, the worker could pull about 114 to 163 kg (250 to 358 pounds).

A famous scene from the tomb of nomarch Djehutihotep at Deir el-Bersha shows transportation of a stone statue of a seated man about 5 m high on a wooden sled (Arnold 1991, 61). One hundred and seventy-two men pull on ropes, while one figure pours a liquid under the sled runners to lubricate them. Arnold estimates that the statue would have weighed about 58 tons, so that each man is pulling about one-third of a ton or about 330 kg, indicating a coefficient of sliding friction of 0.07. Other images of workers dragging statues on sleds, usually accompanied by another person pouring liquid under the runners, are not uncommon in Old Kingdom tomb reliefs (Malek 1986, 65). When pulling a standing life-sized stone statue, only two men are shown on the ropes, and here again calculations of reasonable mass and force indicate the same approximate value of the coefficient of friction. All these depictions may of course be influenced by space available or artistic considerations. Nevertheless, they are all consistent with a coefficient of friction near 0.10. The accuracy of Egyptian art in portraying daily activity suggests that using these scenes as a literal guide is not implausible. Taking all this into account, a coefficient of sliding friction of 0.10 is considered a reasonable estimate of the resistance to dragging sleds on lubricated pathways of transverse smooth wooden beams, and this value was used to compute the work needed to haul stones from the quarry.

The distance from the centre of the base of Khufu's pyramid to the most remote (south) end of the Central Field quarry is about 635 m (Lehner 1984). Given the volume (and hence the mass) of stone moved per day at any time in construction (Table 2 column 2), the distance to haul stones, the coefficient of friction, and the average amount of work provided by each worker each day, it is simple to compute the number of workmen required for hauling the stones. The resulting manpower for hauling stones for each level of the pyramid is listed in column four of Table 2. Again, this is a minimum for this component of transportation. It does not include any extra factor for inefficiencies in technique, nor for the likelihood that the path from the quarries to the construction site was other than perfectly direct. The total manpower required to transport stone from the quarry to the building site at any level is the sum of columns three and four in Table 2, not yet adjusted to account for inefficiencies.

Manpower for transport of stone from the quarries depends directly on the distance travelled and on the coefficient of friction. Reducing the coefficient of friction by a factor of two — easy to achieve by lubricating the paths — reduces by half a large component of the workforce required for pyramid construction. The Egyptians surely recognized the importance of reducing friction. Remote quarries (many kilometres distant) would call for impossibly large numbers of men. We expect that the quarries for the bulk of building material will be found close by every Egyptian solid stone pyramid, probably within one kilometre or less. The need for a satisfactory quarry close by was surely one factor in determining where stone pyramids were built. Future excavations of pyramid complexes should consider the location of nearby quarries.

As an alternative and perhaps more intuitive approach to estimating the manpower required to move stones from the quarry to the installation point on the pyramid, we may compute the energy demands of dragging stones up a simple inclined ramp. Plausible configurations for ramps to construct Khufu's pyramid have been presented in detail by Lehner (1984; 1985) and Dunham (1956). Figure 3 indicates the basic geometry of the situation. Distance D is the distance from the quarry to the pyramid, 635 m in our case. Length H is the height increase from quarry floor to installation point at whatever level the pyramid has reached. The angle of the ramp is α; the distance along the ramp is R. The distance on the flat is A. The values of H, R, and A change as the pyramid rises and the ramp must be extended. The ramp section can have turns in it, and can be separated into two or more segments, but the same energy demand is required. For example, one part of the ramp could lead out of the quarry onto the plateau, and another part could wind around the pyramid.

The total energy demand to drag the stone from the quarry to the top of the pyramid is computed in three parts. The energy to pull the stones over the flat section is the same equation as above, with the present variables, $g\ m\ C_f\ \ A$. The energy required to draw the stone up the ramp has two components. Pulling against friction requires $gm\cos(\alpha)C_f R$, where $gm\cos(\alpha)$ is the force of the stone produced by gravity perpendicular to the surface of the ramp. Lifting the stone against gravity as it goes up the ramp adds the term $gm\sin(\alpha)R$, $g\sin(\alpha)$ giving the component of the gravitational force down the ramp.

The mass m transported per day is determined from the construction rate (Table 2 column two). The energy required for transportation was computed from the three terms. Dividing by the average energy provided by a workman in one day gives the number of men required to move stones from the quarry to the top of the pyramid each day (Table 2 column five). As before, this includes no extra factor for any inefficiency.

Note that the sum of the values in columns three and four of Table 2 exactly equals the value in column five. Both methods of estimating transportation manpower requirements give the same result. The reason can be explained with reference to Figure 3 and to the equations for energy demands using the ramp. The energy demand to pull the stones up the ramp against the component of gravity down-ramp is $gm\sin(\alpha)R$. But $\sin(\alpha)R$ is H, the total height lifted, so the work against gravity is equal to mgH, the same expression used in the first computation, where $H = (h + 19.0)$. The energy demand to pull the stones against friction is $gmC_f A + gm\cos(\alpha)(C_f \times R)$. Now $\cos(\alpha)R = B$, and $A + B = D$, so the two terms become one: $g\ m\ C_f\ D$. This is the equation for the horizontal component of work against friction, where D is, as in the first calculation, the horizontal distance separating quarry and pyramid. The two methods of computing transportation energy demands, by the abstract approach of two independent components, and by an assumed ramp, are mathematically identical.

We also see that the angle α of the ramp drops out of the equations for computing the energy demand of moving stones from the quarry to the construction site via a ramp. Strictly according to the basic principles of physics, the energy demand of using a ramp to deliver stones to a pyramid is independent of ramp angle. Of course, side effects and various practical difficulties may make a steeper ramp more difficult to use. For example, blocks will tend to slide down a ramp whose angle is steeper than 5.7 degrees if the coefficient of friction is 0.10 or less (free sliding can occur where the tangent of the slope is larger than the coefficient of friction). Otherwise ramp angle is irrelevant in estimating transportation energy demands. This is a practical result which is one factor to be considered when discussing ramps.

To make an estimate of actual manpower employed, we must multiply the figure determined (Table 2 column five), which represents an ideal result with no inefficiencies, by some factor to account for

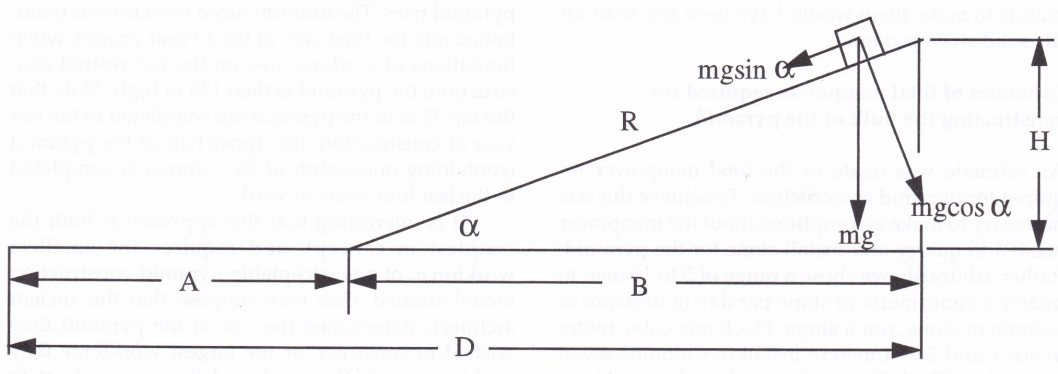

Figure 3. *Geometry of transportation path using a ramp (elevation).*

inefficiencies in technique and to include the likelihood that the distance from the quarry to the pyramid was larger than the direct distance *D*. To some extent the extra distance required is already incorporated, since the distance used (635 m) is that from the pyramid to the most remote point of the quarry (Lehner 1984). Since dragging stones on sleds is very simple and directly applies force to the point required, inefficiencies of technique will not be too large. I have chosen an increase of 50 per cent to arrive at the final estimate of manpower for transportation (Table 2 column six). The manpower for moving stone blocks from the quarry to the point of installation is the largest single factor in pyramid construction; it is always greater than the men required to quarry and install stone in this model. This is true even in the case where the quarry is only some 600 m from the pyramid. Note that manpower for transportation does not vary greatly as the pyramid rises. The pyramid is 70 m high before the manpower for transport varies more than 10 per cent from its initial value at the base. Increasing demand for men for lifting offset the decrease in demand for horizontal transport.

Ramps are often proposed in pyramid construction, and it would be interesting to know what demands their construction would impose. Proposed ramps vary from large free-standing structures whose volume exceeds the volume of the pyramid (Lehner 1984), to small ramps wrapping around the pyramid and standing on unfinished steps of masonry (Dunham 1956). We do not know what kind of ramps were used to build the pyramids, much less the exact dimensions, so estimates of manpower to build ramps are not attempted here. Since ramps were apparently made of loose material — easily gathered, moved, and piled up — the manpower demands to make them would have been less than for the solid stone pyramid.

Estimates of total manpower required for constructing the bulk of the pyramid

An estimate was made of the total manpower required for pyramid construction. To achieve this it is necessary to make assumptions about the manpower needed to quarry and install stone for the pyramid. Rather arbitrarily we chose a range of 5 to 10 men to quarry a cubic metre of stone per day (a measure of volume of stone, not a single block one cubic metre in size), and 2 to 4 men to install it. Columns seven and eight of Table 2 show the resulting low and high estimates of total manpower to build the pyramid, including transportation, quarrying, and installation. The value in column seven is seven times the value in column two plus the value in column six; the values in column eight include fourteen times column two.

Table 2 was developed to illustrate in detail the principles behind estimation of manpower given a construction rate schedule. Every value in Table 2 depends on the choice of construction rate schedule — in this case, that of a linear decrease of construction rate with pyramid height. This is one way to build the pyramid and meet all constraints, though a simpler approach with slightly lower manpower demands is also possible, as follows.

Changing the point of view entirely, one may well ask how many workers are required if a uniform-sized workforce were employed for most of the duration of pyramid construction. Such an approach might have been adopted by the ancient Egyptians. A computer program was used to determine iteratively the construction rate needed at each level to maintain a constant total workforce and to complete the pyramid in the required amount of time. The requirement that the workforce be constant was omitted at the upper levels where working area is restricted. The resulting number of men and construction rate to build Khufu's pyramid in 8400 days are shown in Figure 4. As before, a higher and lower manpower estimate was made, the larger requiring ten men per day to quarry each cubic metre of stone and four to install that amount, and the smaller requiring five men for quarrying and two for installation. The size of the total workforce is 8380 to 10600 men. Table 3 shows details of manpower calculations for the lower manpower estimate. The construction rate drops as men are shifted from quarrying to the increasing demands of lifting stone as the pyramid rises. The uniform-sized workforce is maintained into the final year of the 23 year project, when limitations of working area on the top restrict construction; the pyramid is then 110 m high. Note that the top 35 m of the pyramid are completed in the last year of construction; the upper half of the pyramid (containing one-eighth of its volume) is completed in the last four years of work.

It is interesting that this approach is both the simplest in concept, and requires the smallest workforce, of any acceptable pyramid construction model studied. One may suppose that the ancient architects determined the size of the pyramid they wished to construct, or the largest workforce they could maintain, then gathered the men and set to work. No changes in the number of workmen would

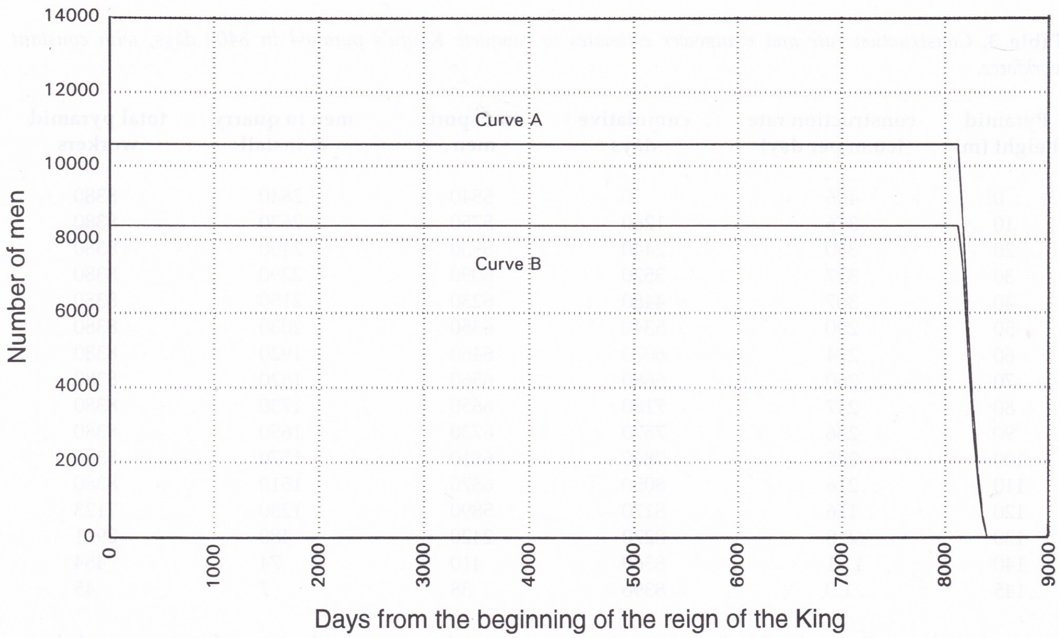

Figure 4a. *Higher and lower estimates of number of men required for construction of the bulk of Khufu's pyramid, with days in the King's reign, which will complete Khufu's pyramid in 8400 working days. (Curve A is higher estimate of 10,600 men; Curve B is lower estimate of 8380 men from Table 3 columns three and six.)*

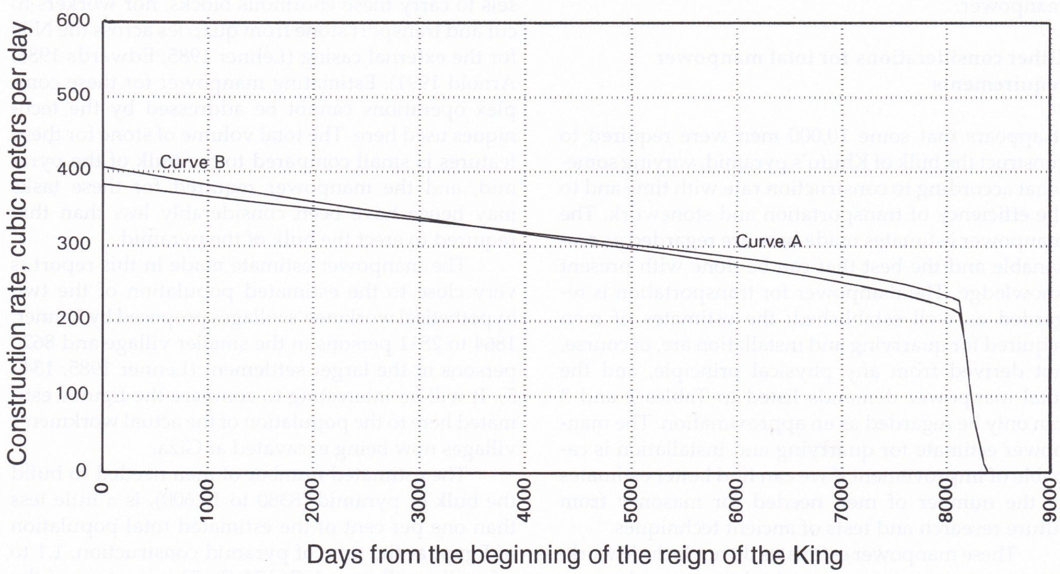

Figure 4b. *Construction rates associated with the higher and lower number of men required for construction of the bulk of Khufu's pyramid, with days in the King's reign. (Curve A is for higher estimate of 10,600 men; Curve B is for lower estimate of 8380 men from Table 3 columns two and three.)*

Table 3. *Construction rate and manpower estimates to complete Khufu's pyramid in 8400 days, with constant workforce.*

Pyramid height (m)	construction rate (cu.m per day)	cumulative days	transport men	men to quarry & install	total pyramid workers
0	406	0	5540	2840	8380
10	376	1260	5750	2630	8380
20	350	2440	5930	2450	8380
30	327	3520	6090	2290	8380
40	307	4480	6230	2150	8380
50	290	5340	6350	2030	8380
60	274	6070	6460	1920	8380
70	260	6680	6560	1820	8380
80	247	7180	6650	1730	8380
90	236	7570	6730	1650	8380
100	225	7860	6810	1570	8380
110	216	8050	6870	1510	8380
120	176	8170	5890	1230	7123
130	68.8	8270	2420	480	2901
140	11.1	8360	410	74	484
145	1.0	8396	38	7	45

be required later. Certainly this is a very simple scheduling of manpower. Managing a constant-sized workforce has numerous advantages, and the approach appears to make the most efficient use of manpower.

Other considerations for total manpower requirements

It appears that some 10,000 men were required to construct the bulk of Khufu's pyramid, varying somewhat according to construction rate with time and to the efficiency of transportation and stonework. The manpower estimates made here are regarded as reasonable and the best that can be done with present knowledge. The manpower for transportation is regarded as well established; the estimates of men required for quarrying and installation are, of course, not derived from any physical principle, and the total manpower demands listed in Tables 2 and 3 can only be regarded as an approximation. The manpower estimate for quarrying and installation is capable of improvement if we can find better estimates of the number of men needed for masonry from future research and tests of ancient techniques.

These manpower estimates do not include workers for constructing ramps, harbours, canals, temples, storage magazines, workshops, bakeries, dwellings for the workers, or for logistical support. Nor do the estimates take into account the number of workers to make the internal passages and chambers of the pyramid (some roofed with 50 ton granite blocks), nor workers to quarry and transport granite from Aswan, nor shipwrights to build the large vessels to carry these enormous blocks, nor workers to cut and transport stone from quarries across the Nile for the external casing (Lehner 1985; Edwards 1988; Arnold 1991). Estimating manpower for these complex operations cannot be addressed by the techniques used here. The total volume of stone for these features is small compared to the bulk of the pyramid, and the manpower required for these tasks may hence have been considerably less than that required to erect the bulk of the pyramid.

The manpower estimate made in this report is very close to the estimated population of the two hypothetical workmen's villages proposed by Lehner: 1864 to 2811 persons in the smaller village and 8620 persons in the larger settlement (Lehner 1985, 134–5). It will be interesting to compare the figures estimated here to the population of the actual workmen's villages now being excavated at Giza.

The estimated number of men needed to build the bulk of pyramid (8380 to 10,600), is a little less than one per cent of the estimated total population of Egypt at the time of pyramid construction, 1.1 to 1.5 million (Butzer 1976, 74–7). This portion of the population was probably a larger organized group than any previous village or palace community. The construction of Khufu's pyramid complex was not the first enterprise of this kind, however, but was preceded by about a century of similar works of increasing complexity. During this century experience in organizing and managing such communities must have accumulated, as well as expertise in construction techniques.

If fewer than 8400 days were worked, higher values for construction rates and manpower are required. For construction times differing only a little from 8400 days, the revised values are simply in proportion to the values of Tables 2 or 3 as the construction time is in proportion to 8400 days. If construction was limited to the 100-day period of the annual inundation of the Nile, the construction time is reduced very substantially to 2300 days. For this case the results were recomputed, using the same parameters previously employed, except for building time. The low- and high-manpower estimate is now 35,800 to 45,600 men.

Figure 5 graphs the manpower estimates for building Khufu's pyramid in 23 years, in 100-day intervals each year, as a function of time through the King's reign. The extreme and episodic changes in manpower raise the question whether the ancient Egyptians would have regarded this as an efficient use of resources, or indeed whether such an approach would even have been manageable. Attempting to save this approach by adopting continuous quarrying, accumulating a supply of stone to be moved and installed during the 100-day intervals, has little effect on the problem of rapid fluctuations in total manpower since transport and installation demand most of the men.

One outcome of the calculations presented here

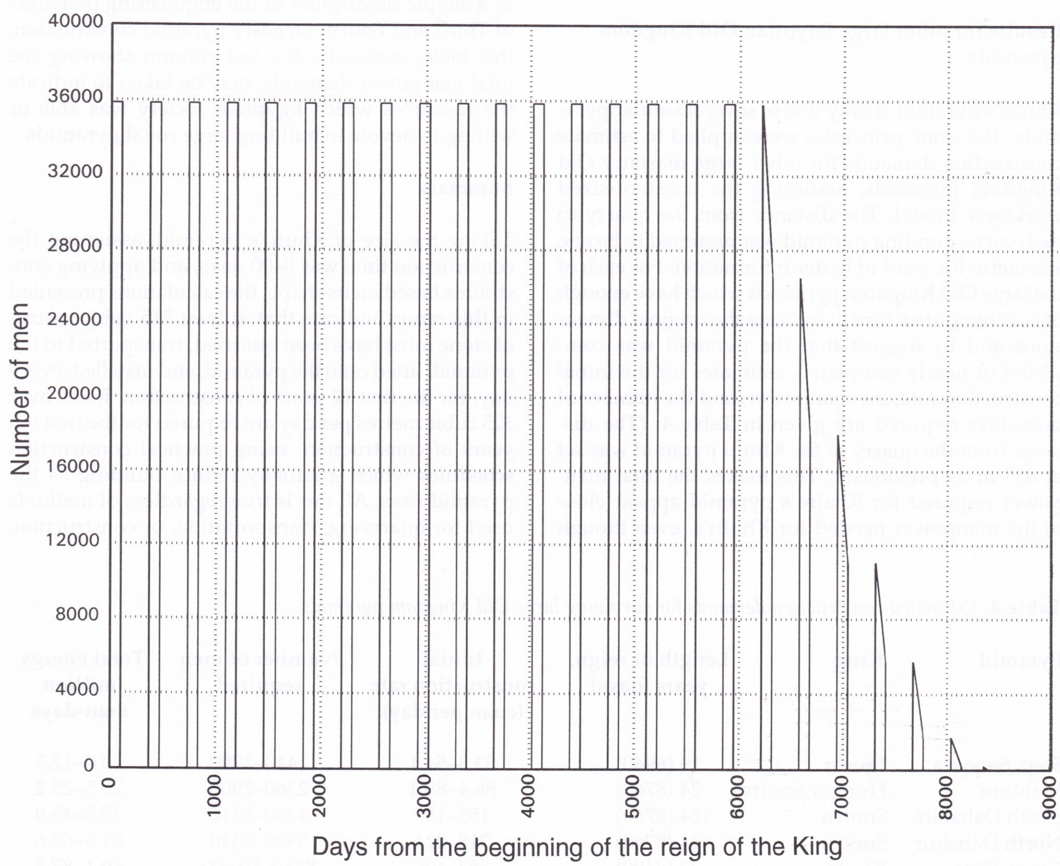

Figure 5. *Manpower schedule to build Khufu's pyramid in 100-day intervals, one interval each year of his reign. This is the lower of two estimates described in the text.*

is an estimate of 69 to 87 million man-days to construct Khufu's pyramid. This is simply a way to express the manpower or energy required to quarry the stone for the pyramids, haul blocks from the quarry to the construction site, lift the blocks, and install them. This result is independent of the variation of construction rate with height, and of the time available to build the pyramid. All schedules for pyramid construction meeting the constraints given here must give the same result. Fewer men would be needed if the quarries were closer to the pyramid than 635 m, if the coefficient of friction were less than 0.10, or if the average density of the pyramid were less than 2.70 kg m^{-3}. More men would be needed if more than 10 men were required to quarry a cubic metre of stone in a day, or more than 4 men to install a cubic metre of stone in a day.

Results for other large Egyptian Old Kingdom pyramids

Khufu's pyramid is only one of several similar pyramids. The same principles were applied to estimate construction demands for other large masonry Old Kingdom pyramids, assuming the constant-sized workforce model. The distance from the quarry to each corresponding pyramid was assumed to be one kilometre for want of better information. For each of the large Old Kingdom pyramids which have enough of surviving structure to indicate the original dimensions and to suggest that the pyramid was completed or nearly completed, estimates for the initial construction rate, the manpower, and the number of man-days required are given in Table 4. (The distance from the quarry to the Khufu pyramid was set at 635 m as previously. This makes the total manpower required for Khafre's pyramid appear close to the manpower needed for Khufu's, even though Khafre's is slightly smaller). The initial (and maximum) construction rates shown are considered to be well estimated, if we accept the ancient indications of the length of each Pharaoh's reign.

The figures in Table 4 show the differences in demands made by construction of each pyramid. These are largely the consequence of differences in size, and to a lesser extent of differences in time available for construction. There is an increase in the effort expended in building each successive pyramid up to Khufu's pyramid. A modest drop in demand for resources applies to Khafre's pyramid, followed by a very large reduction in building effort at Menkaure's pyramid. In terms of construction demands, Menkaure's pyramid is smaller than the Step pyramid, the pyramid which started the sequence of pyramid building over 100 years previously. As well as a simple description of the engineering demands of Third and Fourth Dynasty pyramid construction, this table, especially the last column showing the total manpower demands, may be taken to indicate the resources which Egyptian society was able or willing to devote to building large royal pyramids.

Summary

Taking the size of Khufu's pyramid, assuming the construction time was 8400 days, and applying constraints based on its shape, the calculations presented in this report indicate that at least 315 cubic metres of stone must have been quarried, transported to the pyramid, lifted onto the pyramid, and installed, every day, for the first 12 years of construction. Rates over 325 cubic metres per day are required for the first ten years of construction, using practical construction schedules which gradually reduce building as the pyramid rises. All this is true regardless of methods used for quarrying, transportation, or construction.

Table 4. *Estimated construction demands for surviving large Old Kingdom pyramids.*

Pyramid	King	Length of reign, years (days)	Initial construction rate (cu.m per day)	Number of men required	Total energy, million man-days
Step, Saqqara	Djoser	19 (6940)	53.3–54.1	1440–1790	10.0–12.3
Maidum	Huni or Snofru	24 (8760)	86.4–88.4	2360–2900	20.5–25.2
South Dahshur	Snofru	24 (8760)	155–159	4330–5310	37.5–45.9
North Dahshur	Snofru	24 (8760)	218–224	5980–7340	51.9–63.6
Great, Giza	Khufu	23 (8400)	383–406	8380–10,600	69.1–87.3
Second, Giza	Khafre	26 (9500)	273–287	7660–9370	71.6–87.0
Third, Giza	Menkaure	18 (6570)	39.6–39.6	1060–1340	7.0–8.7

A total manpower requirement of 10,000 men to build the bulk of Khufu's pyramid is derived from the construction rate. Values for the same factors were determined for other large Old Kingdom pyramids.

This report shows that the rate of pyramid construction can be well determined using geometrical and time considerations, without any reference to particular construction techniques. The construction rate estimates can serve as a test for proposed construction techniques. Given the construction rate, the manpower required for moving stone from the quarry to the point of installation can be estimated from simple principles of physics. The construction rate also can be used to compute manpower required for the other major tasks — quarrying and installation. Improved estimates of manpower could be made if we were able to determine how many experienced masons were required to quarry and install a given volume of stone in a day. Better knowledge of the location of the quarries for each pyramid would also help, as would precise figure for the average daily energy expenditure of men hauling stones. Until such further evidence comes to light, the approach outlined here is a good guide to the demands of pyramid construction.

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Acknowledgements

Dr Clark Hamilton, Dr J.M.W. Kooijman, James Lowdermilk, Robert Lowdermilk, Ellen Steiner, and a reviewer made valuable contributions to this investigation through their ideas and discussions. Special thanks are due to Dr Mark Lehner who created and provided the map of Giza. Ellen Steiner carefully reviewed the manuscript and suggested improvements.

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